ALGEBRAS DEFINED BY FINITE GROUPS*

BY

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Introduction.

1. In a paper presented to this Society at a previous meeting, † the general structure of linear associative algebra was discussed and certain fundamental theorems of great generality were proved. The present paper is an application of these theorems to the study of what may be called *group algebras*. By a group algebra is meant that linear algebra whose units are defined to be such that each unit e_i corresponds to an operator O_i of some given abstract finite group, ‡ and conversely, and such that for each equation of the group $O_i O_j = O_k$ corresponds an equation $e_i e_j = e_k$ of the algebra. From the symbolic point of view the algebra differs from the group only in that expressions—for brevity, let us say, numbers—of the form $\Sigma x_i e_i$ are possible, wherein the coefficients x_i are any scalars. § That this algebra is linear and associative, is obvious from the definition. When no confusion is feared, the notations and terminology of the group and of the algebra will be used interchangeably.

2. In the paper cited as *Theory* is developed the theorem that the numbers of a linear associative algebra are subject to the laws of matrices, and certain conclusions are drawn from this fact. This method of development enables us to make immediate use of any theorem needed which is true of matrices, and so saves a redevelopment of many such theorems. In the present paper I shall consider the numbers as multiple algebraic entities, referring to the theory of matrices only when some needed theorem is to be translated into an algebraic theorem.

3. In Part 1 of the paper the general form of any group algebra is to be discussed and certain general theorems established. In Part 2 a few special cases

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† *Theory of Linear Associative Algebra*, Transactions of the American Mathematical Society, vol. 4 (1903), pp. 251–287. This paper is hereafter cited as *Theory*.

‡ It was POINCARÉ, *Sur l'intégration algébrique des équations linéaires et les périodes des intégrales abéliennes*, Journal de Mathématiques, ser. 5, vol. 9 (1903), pp. 139–213, who first made such a correspondence (p. 183)—Cf. also A. YOUNG, Proceedings of the London Mathematical Society, vol. 33 (1900–01), p. 97, where expressions linear and homogeneous in literal substitutions are used as operators.

§ I. e., for the present purposes, ordinary complex numbers.

are considered, in order to elucidate the theory and to furnish comparisons with other well-known methods.* In Part 2, § 1, it is shown that every abelian group of order n defines the same group algebra. Professor E. H. MOORE has pointed out that this result is not equally true for all fields of coefficients. This paper relates only to the scalar continuous field. In general we have the question to study: *With respect to any field of coefficients, what groups determine the same algebra, and what common property do such groups have?* In Part 3 is considered the ultimate connection of this theory with FROBENIUS's theory of group characters and group determinants, BURNSIDE's continuous groups defined by finite groups, and DICKSON's theory of groups in an arbitrary field defined by finite groups. In anticipation as well as summary it may be said that the present theory seems to furnish a desirable common ground for all these theories.

4. The theorem that *every group algebra is semi-simple or matric* (Part 1, § 3) was proved by POINCARÉ, in the article cited,† by connecting the work of MOLIEN and CARTAN ‡ on linear associative algebras from the standpoint of continuous groups, with that of FROBENIUS on the group determinants arising from groups of finite order. On the basis of the *Theory* a direct proof of this theorem is given in Part 1. The general theory of matric algebras will be developed later.

PART 1. GENERAL THEOREMS.

1. For present purposes I proceed to state certain propositions — theorems or obvious deductions from theorems of the general theory.

Since the numbers of a linear associative algebra are subject to the theorems of matrices, it follows that a number satisfies a certain irreducible equation with scalar coefficients — the characteristic equation of the number; the degree of this equation is the degree of the number. — The general number, with indeterminate scalar coefficients x_i, satisfies the characteristic equation of the algebra, a certain irreducible equation, with coefficients polynomials in the indeterminates x_i, with scalar coefficients — the characteristic equation of the algebra;

* The special examples chosen are the algebras derived from *cyclic groups* § 1, *abelian groups* § 1, *the dihedron groups* § 2, *the symmetric groups on three and four letters* §§ 2, 4, *the tetrahedral group* § 3, and *a certain group of order* 16 § 5.

† loc. cit., pp. 184–186.

‡ MOLIEN, *Ueber Systeme höheren complexen Zahlen*, Mathematische Annalen, vol. 41 (1893), p. 83; vol. 42 (1893), p. 308.

CARTAN, *Les groupes bilinéaires et les systèmes de nombres complexes*, Annales de Toulouse, vol. 12 (1898).

The results and methods of MOLIEN and of CARTAN are closely related, although neither CARTAN nor POINCARÉ refers to MOLIEN. In particular, CARTAN's simple systems or p^2-ions, the generalization of quaternions and nonions, (the quadrate algebras of this paper) are MOLIEN's primitive (ursprüngliche) systems.

the degree of this equation is the degree of the algebra; the degree of an algebra in n units is at most n or $n+1$ (according as the algebra has or has not a modulus). It may happen that the degree of every number of an algebra is less than the degree of the algebra itself, every equation resulting from the characteristic equation of the algebra by the substitution of particular scalars for the indeterminate scalar x_i, becoming reducible.—The structure of the algebra depends in a most intimate way upon its characteristic equation.

2. The definition of associativity leads at once to the theorem that if the latent regions of any faciend or right-handed multiplier,—or simply right multiplier,—be determined, then every facient or left multiplier projects each of these regions into itself. Further, the full effect of every left multiplier can be determined when the internal structure of these latent regions is known.

3. The units and all numbers of the algebra are expressible linearly in terms of certain ideal or auxiliary units called *associative units*, represented in the *Theory* p. 261, by λ_{ijk}. An associative unit may often itself be a unit of the algebra. This happens in the case of group algebras. The third subscript, k, is the *weight* of the associative unit. The aggregate of numbers of which each term has the weight zero form a sub-algebra, since the weight of a product is the sum of the weights of its factors. A sub-algebra of this kind was called by CARTAN * *semi-simple*, his case being such that if λ_{ij0} belongs to it, then λ_{ji0} also belongs. The present writer prefers to call such an algebra or sub-algebra *matric*. It is to be shown that *a group algebra is matric*, that is, its units may all be taken to be of the form λ_{rs0}, and such that if λ_{rs0} is a unit, λ_{sr0} is also a unit. The converse is not true, as may be seen by considering that a matrix of order n, which may be represented as a *quadrate* algebra, does not determine a group of order n^2. †

The immediate problem will now be taken up.

4. *If, in any algebra, a region* $\{\theta_i\}$ *is such that for each number belonging to the region we have the equation* $\theta_i(\psi - g_i) = 0$, ψ *being a given number, then the region* $\{\theta_i\}$ *is free from semilatent regions of* ψ, *and the form of any left number of the algebra* ϕ_i *is* $\Sigma a_{rs0}\lambda_{rs0}$ *plus terms which annul* $\{\theta_i\}$.

For, in this case the integers ‡ m_r, m_s, etc., all become $=1$. Hence the weight t must be zero to satisfy the conditions $t < m_r$, $t \equiv m_r - m_s$.

5. *If, for one number* ϕ'_i *of the region* $\{\theta_i\}$, *which has* m *dimensions, let us suppose, we have* $\theta_i\phi'_i = \theta_i$ *whatever number of that region* θ_i *may be; and if all the numbers* $\phi''_i, \cdots, \phi^{(m)}_i$, *independent of* ϕ'_i *and each other, but belong-*

* *Les Groupes Bilinéaires et les Systèmes de Nombres Complexes*, Annales de la Faculté de Sciences de Toulouse, vol. 12 (1898), p. 57.

† At least a group in the ordinary sense. If the definition of group were extended so as to include cases in which the product of any two operators gave a third operator multiplied by a scalar factor, the case would be different. A matrix does determine a group of order n^3, however.

‡ *Theory*, p. 265, § 4, et seq., p. 273, § 6.

ing to the region, give $\theta_i \phi_i^{(r)} = 0$ $(r = 2 \cdots m)$, *then the algebra of that region is definable by the units* λ_{r10}, $r = 1, \cdots, m$, $\phi_i^{(r)} = \lambda_{rio}$.

6. If there are other regions of m dimensions of similar character, then each is definable by m units $\lambda_{q_i ro} (i = 1, \cdots, m)$. If any one of the first set can multiply one of this set giving a product which does not vanish, as

$$\lambda_{t10} \cdot \lambda_{q_i ro} = \lambda_{tro},$$

then $\lambda_{q_i ro}$ must have the form λ_{1ro}, and by multiplication of the two sets we find the second set must be representable by $\lambda_{sro} (s = 1, \cdots, m)$. If there are m numbers of the form λ_{iio} each determining a region thus, the entire set of numbers so determined is representable by $\lambda_{tuo} (t, u = 1, 2, \cdots, m)$. There are then m^2 such units λ, forming a quadrate algebra.

7. From the properties of the operators of the group, we have for any analogous unit of the algebra, as e_r, whose $m + 1$ power equals itself, the equations:

$$e_r^m \cdot a = a = a \; e_r^m.$$

In this equation a is any number of the algebra. Therefore as a multiplier, e_r has a characteristic equation

$$(e_r^m - 1) \cdot = 0 = \cdot (e_r^m - 1).$$

If ω is a primitive mth root of unity, this equation may be written in the form

$$(e_r - 1)(e_r - \omega) \cdots (e_r - \omega^{m-1}) \cdot = 0.$$

Since the ground is of order $n = qm$, and since this equation contains each factor only to the first degree, it follows e_r *has* n *latent roots in* m *sets of* q *identical roots, and each distinct root has a latent region of* q *dimensions and no semilatent* regions of any species.*

8. It follows that if we construct the number

$$\begin{aligned} f_r &= (e_1 - 1)(e_1 - \omega^{-1}) \cdots (e_1 - \omega^{-(r-1)})(e_1 - \omega^{-(r+1)}) \cdots (e_1 - \omega) \\ &= 1 + \omega^r e_1 + \omega^{2r} e_1^2 + \cdots + \omega^{(m-1)} e_1^{m-1}, \end{aligned}$$

it will give us, when used as a right multiplier on all numbers of the algebra, the region $\{ c_r \}$, for every number of which,

$$c_r \cdot f_r = c_r, \qquad c_r \cdot f_s = 0 \qquad (r \neq s),$$

and also

$$c_r \cdot e_1 = \omega^r c_r.$$

The m numbers $f_1, f_2, \cdots, f_m$ will thus serve to divide the ground into m

* WHITEHEAD, *Universal Algebra*, vol. 1, p. 258. For a higher type of semi-latency compare *Theory*, p. 271–272.

exclusive regions, each corresponding to a root of e_1. If the numbers f were used as left multipliers, the same would be true, although in general the regions would be made up differently. For any operator of the group a similar statement holds.

9. Let a set of independent generators of a group be $e_1, e_2, \cdots, e_c$. Starting with any one of the generators, say e_1, of period or multiplicity m_1, $e_1^{m_1} - 1 = 0$, and putting ω for a primitive m_1th root of unity, let us form the numbers $f_{11}, f_{12}, \cdots, f_{1m_1}$, which give us the m_1 distinct latent regions of $(\ \ )\cdot e_1$. For convenience, let us call the number for any multiplier whatever which annuls the whole ground of the algebra except the latent region of a certain root g of the multiplier, a *latent* of the multiplier, corresponding to the root g. By a theorem in matrices* the multiplier is the sum of the products of the roots into their corresponding *latents*, when there are no semi-latent regions. We may then call $f_{11}, \cdots, f_{1m_1}$ the latents of e_1. Each is of the form

$$f_{1r} = \frac{1}{m_1}(1 + \omega^r e_1 + \omega^{2r} e_1^2 + \cdots + \omega^{(m_1-1)r} e_1^{m_1-1}) \quad (r = 1, 2, \cdots, m_1).$$

These numbers are the products of all the latent factors except the one corresponding to the root ω^{-r}, divided by the corresponding factors with ω^{-r} substituted for e_r. These latents of e_1 divide the whole ground of the algebra into regions $\{\theta_r\}$, such that for any number in one of these regions, as θ_r,

$$\theta_r \cdot f_{1r} = \theta_r,$$

and

$$\theta_r \cdot f_{1s} = 0 \qquad (s \neq r).$$

Necessarily $f_{1r}\cdot$ is itself included in the region $\{\theta_r\}$, for

$$f_{1r} \cdot f_{1r} = f_{1r}.$$

Any number of the region $\{\theta_r\}$ must have the form

$$pf_{1r},$$

where p is some product of the generators of the group. Since the region $\{\theta_r\}$ is of n/m_1 dimensions, there are n/m_1 independent numbers of the form just given, and as $e_2\phi$ does not vanish for any number of the algebra, $e_2 f_{1r}$, as also $e_3 f_{1r}$, $e_2 e_3 f_{1r}$, $e_3 e_2 f_{1r}$, etc., are all to be found among the numbers of $\{\theta_r\}$. Any number of the algebra as left multiplier must project these regions into themselves,† so that we have for any number ϕ and any region $\{\theta_r\}$,

$$\phi \cdot \{\theta_r\} = \{\theta_r\}.$$

We may then state the following theorem:

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* TABER, *Theory of Matrices*, American Journal of Mathematics, vol. 12 (1890), p. 378.

† *Theory*, p. 268.

A first separation of the numbers of the algebra into mutually exclusive regions is effected by multiplying the entire ground of the algebra on the right by the latents of $(\ )\cdot e_1$, *these latents being numbers of the algebra depending only on the generator* e_1. *Every number of the algebra as a left multiplier is equivalent to the sum of partial multipliers in terms of* λ_{ijk}, *corresponding to these regions and each in effect operative only on its corresponding region.*

10. Suppose next that the n independent numbers just found are taken as defining the ground. Then for any one of them, as $e_2 f_{1r}$, we must have, within the region to which $e_2 f_{1r}$ belongs, as many roots, equal or unequal, as the region has dimensions. Let the latents of $(\ )\ e_2 f_{1r}$ be formed, represented by f_{1rs}. The regions outside $\{\theta_r\}$, as $\{\theta_t\}$ give $\{\theta_t\} f_{1rs} = 0$.* Then the numbers f_{1rs} divide $\{\theta_r\}$ into mutually exclusive regions, just as f_{1r} did the whole ground. Letting r and s take all values which give independent numbers, we have the whole ground of the algebra divided more minutely into regions which are exclusive, and also for any one of which, say $\{\theta_{rs}\}$, we have, ϕ being any number of the algebra,

$$\phi\{\theta_{rs}\} = \{\theta_{rs}\}.$$

It is to be noted that the number of distinct roots of $e_2 f_{1r}$ may differ from that of $e_2 f_{1r'}$.

It is also evident that $\phi(\ )$ must be made up of multipliers (possibly ideal only and expressible only in terms of the λ's) each operative on a single region $\{\theta_{rs}\}$, and also, in case $e_2 f_{1r}$ have roots with semi-latent regions (which, it will appear later, it has not), so constituted as to be still commutative with $(\ )\cdot e_2 f_{1r}$. This being the case, $\phi\cdot(\ )$ will be necessarily commutative with $(\ )\cdot e_2$, since it is commutative with $(\ ) e_1$ and $(\ ) f_{1r}$. We have therefore the theorem:

The latents of $(\ ) e_2 f_{1r}$, $(r = 1, 2, \cdots m_1)$, *give a second separation of the ground into exclusive regions, and any number* ϕ *is the sum of multipliers operative only on these regions.*

11. In the region $\{\theta_{rs}\}$ there must be an expression $e_3 f_{1rs}$. If we form the latents of $(\ ) e_3 f_{1rs}$, let them be f_{1rst}; then these expressions will divide the ground of the algebra into exclusive regions as before. We may so proceed with each generator, building up latents for each combination, each set of latents dividing the whole ground into smaller regions.

A repetition of the process of separation for a set of independent generators produces a separation of the ground into mutually exclusive regions, and any facient number of the ground is the sum of parts each operative in only one of these regions.

It is to be noted that this process may be applied to any algebra defined by means of generators, whatever the equations of the generators.

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*Since $(\ ) e_2 f_{1r}$ projects all such regions into $\{\theta_r\}$.

12. Now if any left multiplier ϕ projects the latent regions of $(\ )e_1$ into themselves, and those of $(\ )e_2 f_{1r}$ into themselves, then ϕ projects the latent regions of $(\ )e_2$ into themselves, and likewise those of $(\ )e_2^x e_1^y$ into themselves. Hence when we have used each independent generator in the manner indicated, we shall have ϕ so constituted in terms of the λ's that it projects each latent region of each generator used as a right multiplier into itself, and also projects the right latent regions of any combination of such generators into themselves.* Since no further conditions are imposed then on the associative units, it follows that any new latent would simply give us the same regions. In fact if we had started with more generators than were independent, we should discover this fact by reaching eventually a division into regions $\{E_{qr}\}$, each of which is such that any expression chosen from it has the entire region $\{E_{qr}\}$ for the latent region of its one latent root.† These ultimate regions are reached when we have built up the latents so as to include each generator. Let any such region be $\{E_{qr}\}$ determined by $(\ )F_r$, as above. The totality of the defining numbers of all these regions also define the algebra. $\{E_{qr}\}$ cannot vanish under right-hand multiplication by every number, but for some one at least must give itself as result. This multiplier is $(\ )\cdot F_r$. Also F_r is one of the numbers of $\{E_{qr}\}$, since at least one number of $\{E_{qr}\}$ must give $\{E_{qr}\}$ as the result of right multiplication into the whole domain (otherwise the process would not have terminated).

If now $\cdot F$ is any latent with properly chosen scalar coefficients we know from the theory of matrices that $(\cdot F)^2 = \cdot F$. Hence since $\cdot F^2 - \cdot F = 0$, and since F has but one root in its own region, having the root zero everywhere else, it follows that for each number E_{qr}, we must have for F_r the equation

$$E_{qr}\cdot F_r = E_{qr}.$$

Also $E_{qs}\cdot F_r = 0$, and $F_r^2\cdot = F_r$.

From this equation, and because any left multiplier leaves each region invariant, it follows that in any region of m dimensions we must have m axes of $F_r(\ )$, and these may be taken to be the m numbers E_{qr} defining the region. Hence

$$F_r\cdot E_{qr} = E_{qr} \text{ or } 0.$$

Similarly for any other region E_{pt}:

$$F_r\cdot E_{pt} = E_{pt} \text{ or } 0.$$

If E_{pt} is a latent axis of F_r it cannot be a latent axis of any other latent F_s, for if $F_r E_{pt} = F_{pt}$, then $F_s E_{pt} = F_s F_r E_{pt} = 0$.

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* *Theory*, p. 273, § 6.

† *Example:* Part 2, § 3. The regions E are (λ_{11}), (λ_{22}), (λ_{33}), $(\lambda_{44}, \lambda_{54}, \lambda_{64})$, $(\lambda_{45}, \lambda_{55}, \lambda_{65})$, $(\lambda_{46}, \lambda_{56}, \lambda_{66})$. If $e_3 = e_1 e_2$ appeared as generator also, $e_3\lambda_{qr}$ is already in $\{\lambda_{qr}\}$.

The non-zero-axes of each latent F_r, are zero-axes of every other latent F_s, and the numbers E_{qr} can be idemfactorial only for one latent on the left F_q, and one on the right F_r, or $E_{qr} = F_q E_{qr} F_r$.

13. The only numbers E which can have unity for root must be of the form $E_{rr} = F_r$. There can be only one in each region; for, if two, let the second be E'_{rr}. Then $E_{rr} \cdot E_{rr} = E_{rr}$, $E_{rr} \cdot E'_{rr} = E_{rr}$, therefore

$$E_{rr}(E_{rr} - E'_{rr}) = F_r(E_{rr} - E'_{rr}) = E_{rr} - E'_{rr} = 0.$$

Hence $E'_{rr} = E_{rr}$.

Each region therefore is a case of the theorem of § 5 above, and the defining elements may be written in the form

$$\lambda_{qr0}.$$

Since $F_r \cdot$ has the form E_{rr}, $F_r = \lambda_{rr0}$.

14. Again, since $F_r \cdot (\ \ )$ has only one latent axis for the root unity in any region, it follows that in some other region there are one or more such axes. Hence one such must have the form

$$\lambda_{rs0},$$

and as all the λ's of form λ_{qr0} will multiply this, we find that for each form in λ_{qr0} there is a form in λ_{ts0}. Hence also λ_{sr0} must appear among the forms λ_{qr0}. Therefore there is no second form λ'_{rs0} multiplicable by λ_{rr0}, else it would give two forms λ_{ss0} and λ'_{ss0} which is impossible. Further there is no form λ_{ts0} not included in the forms $\lambda_{qr0} \cdot \lambda_{rs0}$, else it would give $\lambda_{ts0} \cdot \lambda_{sr0} = \lambda_{tr0}$, and there is no such form. Finally, therefore, each latent axis of $F_r = \lambda_{rr0}$, corresponding to the root unity, lies in a different region $\{E_{vt}\}$, and hence the regions combine according to theorem of § 6. We may then state the final theorem of the investigation:

THEOREM. *The numbers of the algebra are linearly expressible in terms of a set of semi-simple terms of the form λ_{qro}, which are determined by operating on every number of the algebra by the latents $F_q \cdot (\ )$, $(\ ) F_r$. The scalar coefficients of the products must be determined so as to satisfy the equations $\lambda_{qro} \cdot \lambda_{rso} = \lambda_{qso}$. These terms collect in sets of $\mu_1^2, \mu_2^2, \cdots, \mu_p^2$, where $n = \mu_1^2 + \mu_2^2 + \cdots + \mu_p^2$. Each set determines an independent quadrate algebra, the group-algebra being the sum of these sub-algebras, that is, a semi-simple or matric algebra. The general equation of any number of the algebra,*

$$\phi = \Sigma x_{ij} \lambda_{ijo},$$

is

$$|x_{r_1 s_1} - \phi| \cdot |x_{r_2 s_2} - \phi| \cdot \cdots = 0,$$

the μ_i^2 elements of the i-th determinant being made up of the x's of the i-th sub-algebra.

15.* It is evident, as POINCARÉ pointed out,† that, since the operators of the group may be put into s mutually exclusive conjugate classes, if we consider the corresponding numbers of the algebra, then σ_a, the sum of all numbers in the a conjugate class, must be commutative with every number of the algebra. Hence σ_a must take the form

$$\sigma_a = \Sigma \cdot a_j \lambda^{(j)}_{ii0} \qquad (i=1, \cdots, \mu_i;\ j=1, \cdots, s).$$

Since these sums σ are independent of each other, if we multiply them by proper scalars and add them, we can find s independent numbers of the algebra which are commutative with every number of the algebra. *Each of these numbers* ϕ_k, $k = 1, \cdots, s$, *must be of the form* $\Sigma\lambda^{(k)}_{ii0}$. For, evidently $.\Sigma^{(k)}_{\lambda ii0}$ is commutative with every number, and hence is linearly expressible in terms of the σ's; and no number $\Sigma\lambda^{(k)}_{ii0}$ can be broken up in the process of addition used. Therefore, $s = p$, or *there are as many quadrate algebras as there are conjugate classes.* The numbers σ form an abelian algebra.

16. The numbers ϕ_k are the numbers which satisfy the equations

$$\sigma_a \phi_k = g \cdot \phi_k \qquad (a=1, \cdots, p),$$

where g is some scalar coefficient. That is, they are the common latent regions of all the σ's. Also since $\phi_k = \Sigma\lambda^{(k)}_{ii0}$, we may isolate the k-th quadrate algebra by operating on every number of the algebra by ϕ_k; the results will be linearly expressible in terms of μ_k^2 independent numbers, and these numbers will readily yield the units $\lambda^{(k)}_{ij0}$, of the quadrate. This process reduces the labor of computation decidedly.

17. Further, the number μ_k^2 is the multiplicity of the root unity of ϕ_k, so that if n_k is the nullity of the number ϕ_k, its general equation must be

$$(\phi_k - 1)^{\mu_k^2} \phi_k^{n_k} = 0 \qquad (\mu_k^2 + n_k = n).$$

PART 2. APPLICATION TO CERTAIN GROUPS.

1. *Abelian Groups.* An abelian group gives terms of only one form λ_{rr0}. For if any term λ_{rs0} existed, there would be expressions not commutative with all other expressions. It follows that *every abelian group of order n gives the same group algebra.*‡

As a particular case, consider the algebra of the abelian group defined by the generators e_1, e_2, subject to the equations $e_1^4 = 1 = e_2^2$, $e_1 e_2 = e_2 e_1$. Let ω be a primitive fourth root of unity. Let

$$f_{1i} = \tfrac{1}{4}(1 + \omega^{i-1} e_1 + \omega^{2(i-1)} e_1^2 + \omega^{3(i-1)} e_1^3) \qquad (i=1, 2, 3, 4).$$

* §§ 15, 16, 17 were added to the paper May 30, 1904.

† Loc. cit.

‡ It should be remembered that the field of coefficients is the scalar continuous field of numbers.

Whence $e_2 f_{11} = \frac{1}{4} e_2(1 + e_1 + e_1^2 + e_1^3)$, $(e_2 f_{11})^2 = f_{11}$, $(e_2 f_{11})^3 = (e_2 f_{11})$. Therefore the equation of $e_2 f_{11}$ is

$$(e_2 f_{11})(e_2 f_{11} - 1)(e_2 f_{11} + 1) = 0.$$

This gives, with similar equations for $e_2 f_{1i}$,

$$F_{2i} = \tfrac{1}{2}(f_{1i} - e_2 f_{1i}), \qquad F_{2i-1} = \tfrac{1}{2}(f_{1i} + e_2 f_{1i}) \quad (i = 1, 2, 3, 4).$$

Hence the algebra is defined by

$$\lambda_{rr} \qquad (r = 1, \cdots, 8).$$

We find easily

$$e_1 = \lambda_{11} + \lambda_{22} + \omega^3 \lambda_{33} + \omega^3 \lambda_{44} + \omega^2 \lambda_{55} + \omega^2 \lambda_{66} + \omega \lambda_{77} + \omega \lambda_{88},$$
$$e_2 = \lambda_{11} - \lambda_{22} + \lambda_{33} - \lambda_{44} + \lambda_{55} - \lambda_{66} + \lambda_{77} - \lambda_{88}.$$

The correspondence between the forms for this group of order 8 and the cyclic group of order 8, is interesting. If σ is a primitive 8th root of unity, the generator of the cyclic group is

$$e = \Sigma \sigma^{r-1} \lambda_{rr} \qquad (r = 1, \cdots, 8).$$

Whence

$$e = \tfrac{1}{2}[1 + \sigma + e_2(1 - \sigma)] e_1^3,$$

and conversely

$$e_2 = e^4, \qquad e_1 = \tfrac{1}{2}[1 - \sigma + e^4(1 + \sigma)] e^3.$$

Of course the groups are distinct, although the group algebras are identical.

2. *The Dihedron Groups.* The generators are e_1, e_2, with equations $e_1^m = 1 = e_2^2$, $e_2 e_1 = e_1^{m-1} e_2$. Let $\omega^m = 1$. There are two cases, according as m is (1) odd, (2) even:

In the *first case*, let

$$f_{1i} = \frac{1}{m}(1 + \omega^{i-1} e_1 + \omega^{2(i-1)} e_1^2 + \cdots + \omega^{-(i-1)} e_1^{-1}) \quad (i = 1, \cdots, m).$$

Hence

$$F_1 = \tfrac{1}{2}(f_{11} + e_2 f_{11}), \qquad F_2 = \tfrac{1}{2}(f_{11} - e_2 f_{11}),$$
$$F_{2+r} = f_{1r+1} \qquad (r = 1, \cdots, m-1)$$

The forms are therefore

$$\lambda_{11}, \lambda_{22}, \lambda_{2r-1, 2r-1}, \lambda_{2r-1, 2r}, \lambda_{2r, 2r-1}, \lambda_{2r, 2r} \quad (r = 2, \cdots, (m+1)/2).$$

Whence

$$e_1 = \lambda_{11} + \lambda_{22} + \omega^{-1} \lambda_{33} + \omega \lambda_{44} + \omega^{-2} \lambda_{55} + \omega^2 \lambda_{66} + \cdots + \omega^{-\frac{m-1}{2}} \lambda_{mm} + \omega^{\frac{m-1}{2}} \lambda_{m+1, m+1},$$
$$e_2 = \lambda_{11} - \lambda_{22} + \lambda_{34} + \lambda_{43} + \lambda_{56} + \lambda_{65} + \cdots + \lambda_{mm+1} + \lambda_{m+1\,m}.$$

In the *second case*, we find the algebra to be defined by

$$\lambda_{11}, \lambda_{22}, \lambda_{33}, \lambda_{44}, \lambda_{2r-1, 2r-1}, \lambda_{2r-1, 2r}, \lambda_{2r, 2r-1}, \lambda_{2r, 2r} \quad (r = 3, \cdots, m/2 + 1).$$

These give

$$e_1 = \lambda_{11} + \lambda_{22} - \lambda_{33} - \lambda_{44} + \Sigma\omega^{-(r-2)}\lambda_{2r-1,\,2r-1} + \Sigma\omega^{r-2}\lambda_{2r,\,2r},$$
$$e_2 = \lambda_{11} - \lambda_{22} + \lambda_{33} - \lambda_{44} + \Sigma(\lambda_{2r-1,\,2r} + \lambda_{2r,\,2r-1}).$$

3. *The Tetrahedral Group of Order* 12. The generators are e_1, e_2, with the equations $e_1^3 = 1$, $e_2^2 = 1$, $e_2 e_1^2 = e_1 e_2 e_1 e_2$. Let $\omega^3 = 1$. Then

$$f_{11} = \tfrac{1}{3}(1 + e_1 + e_1^2), \quad f_{12} = \tfrac{1}{3}(1 + \omega e_1 + \omega^2 e_1^2), \quad f_{13} = \tfrac{1}{3}(1 + \omega^2 e_1 + \omega e_1^2).$$

The equations of $e_2 f_{11}$, $e_2 f_{12}$, $e_2 f_{13}$, are

$$e_2 f_{11}(e_2 f_{11} - 1)(e_2 f_{11} + \tfrac{1}{3}) = 0, \qquad e_2 f_{12}(e_2 f_{12} - 1)(e_2 f_{12} + \tfrac{1}{3}) = 0,$$
$$e_2 f_{13}(e_2 f_{13} - 1)(e_2 f_{13} + \tfrac{1}{3}) = 0.$$

Therefore we have

$$\lambda_{11} = \tfrac{1}{4}(f_{11} + 3f_{11}e_2 f_{11}), \quad \lambda_{22} = \tfrac{1}{4}(f_{12} + 3f_{12}e_2 f_{12}), \quad \lambda_{33} = \tfrac{1}{4}(f_{13} + 3f_{13}e_2 f_{13}),$$
$$\lambda_{44} = \tfrac{3}{4}(f_{11} - f_{11}e_2 f_{11}), \quad \lambda_{54} = \tfrac{3}{2} f_{12}e_2 f_{11}, \quad \lambda_{64} = \tfrac{3}{2} f_{13}e_2 f_{11},$$
$$\lambda_{45} = \tfrac{3}{2} f_{11}e_2 f_{12}, \quad \lambda_{55} = \tfrac{3}{4}(f_{12} - f_{12}e_2 f_{12}), \quad \lambda_{65} = \tfrac{3}{2} f_{13}e_2 f_{12},$$
$$\lambda_{46} = \tfrac{3}{2} f_{11}e_2 f_{13}, \quad \lambda_{56} = \tfrac{3}{2} f_{12}e_2 f_{13}, \quad \lambda_{66} = \tfrac{3}{4}(f_{13} - f_{13}e_2 f_{13}),$$

the generators are therefore

$$e_1 = \lambda_{11} + \omega^2\lambda_{22} + \omega\lambda_{33} + \lambda_{44} + \omega^2\lambda_{55} + \omega\lambda_{66},$$
$$e_2 = \lambda_{11} + \lambda_{22} + \lambda_{33} - \tfrac{1}{3}\lambda_{44} + \tfrac{2}{3}\lambda_{54} + \tfrac{2}{3}\lambda_{64} + \tfrac{2}{3}\lambda_{45} - \tfrac{1}{3}\lambda_{55} + \tfrac{2}{3}\lambda_{65} + \tfrac{2}{3}\lambda_{46} + \tfrac{2}{3}\lambda_{56} - \tfrac{1}{3}\lambda_{66}.$$

4. *The Octahedral Group of Order* 24. The generators are e_1, e_2, with the equations $e_1^4 = 1 = e_2^3$, $e_2^2 = e_1 e_2 e_1$. Let $\omega^4 = 1$, and put

$$\begin{aligned} e_{r2r} &= f_{1r} e_2 f_{1r} \\ &= \tfrac{1}{16}(1 + \omega^{r-1}e_1 + \omega^{2(r-1)}e_1^2 + \omega^{3(r-1)}e_1^3)\,e_2\,(1 + \omega^{r-1}e_1 + \omega^{2(r-1)}e_1^2 + \omega^{3(r-1)}e_1^3), \\ e_{r2s} &= f_{1r} e_2 f_{1s} \qquad (r, s = 1, 2, 3, 4) \end{aligned}$$

Then the equations of e_{121}, e_{222}, e_{323}, e_{424}, are

$$e_{121}^2(e_{121} - 1)(2e_{121} + 1) = 0, \qquad e_{222}(2e_{222} + \omega)(2e_{222} - \omega) = 0,$$
$$e_{323}^2(e_{323} - 1)(2e_{323} + 1) = 0, \qquad e_{424}(2e_{424} + \omega)(2e_{424} - \omega) = 0,$$

whence

$$\lambda_{11} = \tfrac{1}{3}(2\cdot e_{121}^3 + e_{121}^2), \qquad \lambda_{33} = \tfrac{8}{3}(e_{121}^3 - e_{121}^2), \qquad \lambda_{34} = \frac{2}{\sqrt{-3}}\,e_{321},$$

$$\lambda_{22} = \tfrac{1}{3}(2\cdot e_{323}^3 + e_{323}^2), \qquad \lambda_{44} = \tfrac{8}{3}(-e_{323}^3 + e_{323}^2), \qquad \lambda_{43} = \frac{2}{\sqrt{-3}}\,e_{123},$$

$$\lambda_{55}=f_{11}+e_{121}-2e_{121}^2,\quad \lambda_{56}=(1-\omega)e_{122},\quad \lambda_{57}=(1+\omega)e_{124},$$
$$\lambda_{65}=(1-\omega)e_{221},\quad \lambda_{66}=\omega e_{222}-2e_{222}^2,\quad \lambda_{67}=2e_{221}e_{124},$$
$$\lambda_{75}=(1+\omega)e_{421},\quad \lambda_{76}=2e_{421}e_{122},\quad \lambda_{77}=-\omega e_{424}-2e_{424}^2,$$
$$\lambda_{88}=f_{13}+e_{323}-2e_{323}^2,\quad \lambda_{89}=(1+\omega)e_{322},\quad \lambda_{8a}=(1-\omega)e_{324},$$
$$\lambda_{98}=(1+\omega)e_{223},\quad \lambda_{99}=-e_{222}-2e_{222}^2,\quad \lambda_{9a}=2e_{223}e_{324},$$
$$\lambda_{a8}=(1-\omega)e_{423},\quad \lambda_{a9}=2e_{423}e_{322},\quad \lambda_{aa}=\omega e_{424}-e_{424}^2.$$

These give for the generators

$$e_1=\lambda_{11}-\lambda_{22}+\lambda_{33}-\lambda_{44}+\lambda_{55}+\omega^3\lambda_{66}+\omega\lambda_{77}-\omega\lambda_{88}+\omega^3\lambda_{99}+\omega\lambda_{aa},$$
$$\begin{aligned}e_2=\lambda_{11}+\lambda_{22}-\tfrac{1}{2}\lambda_{33}+\tfrac{1}{2}\sqrt{-3}\,\lambda_{34}+\tfrac{1}{2}(1+\omega)\lambda_{56}+\tfrac{1}{2}(1-\omega)\lambda_{57}+\tfrac{1}{2}(1-\omega)\lambda_{89}\\ +\tfrac{1}{2}(1+\omega)\lambda_{8a}+\tfrac{1}{2}\sqrt{-3}\,\lambda_{43}-\tfrac{1}{2}\lambda_{44}+\tfrac{1}{2}(1+\omega)\lambda_{65}-\tfrac{1}{2}\omega\lambda_{66}+\tfrac{1}{2}\lambda_{67}\\ +\tfrac{1}{2}(1-\omega)\lambda_{98}+\tfrac{1}{2}\omega\lambda_{99}+\tfrac{1}{2}\lambda_{9a}+\tfrac{1}{2}(1-\omega)\lambda_{75}+\tfrac{1}{2}\lambda_{76}+\tfrac{1}{2}\omega\lambda_{77}\\ +\tfrac{1}{2}(1+\omega)\lambda_{a8}+\tfrac{1}{2}\lambda_{a9}-\tfrac{1}{2}\omega\lambda_{aa}.\end{aligned}$$

5. *A group of order* 16, defined by the generators e_1, e_2, e_3, with the equations

$$1=e_1^4=e_3^2,\qquad e_1^2=e_2^2,\qquad e_1e_2=e_2e_1,\qquad e_1e_3=e_3e_1^3,\qquad e_2e_3=e_3e_2^3.$$

Let $\omega^4=1$. The equations of

$$e_{121}=f_{11}e_2f_{11},\qquad e_{222}=f_{12}e_2f_{12},\qquad e_{323}=f_{13}e_2f_{13},\qquad e_{424}=f_{14}e_2f_{14},$$

are

$$(e_{121}^2-f_{11})=0,\qquad (e_{222}^2+f_{12})=0,\qquad (e_{323}^2-f_{13})=0,\qquad (e_{424}^2+f_{14})=0.$$

Hence

$$f_{111}=\tfrac{1}{2}(f_{11}+e_{121}),\qquad f_{121}=\tfrac{1}{2}(f_{11}-e_{121}),\qquad f_{212}=\tfrac{1}{2}(f_{12}-\omega e_{222}),$$
$$f_{222}=\tfrac{1}{2}(f_{12}+\omega e_{222}),\qquad f_{313}=\tfrac{1}{2}(f_{13}+e_{323}),\qquad f_{323}=\tfrac{1}{2}(f_{13}-e_{323}),$$
$$f_{414}=\tfrac{1}{2}(f_{14}+\omega e_{424}),\qquad f_{424}=\tfrac{1}{2}(f_{14}-\omega e_{424}),$$

setting

$$e_{11311}=f_{111}e_3f_{111},\qquad e_{12312}=f_{121}e_3f_{121},\ \text{etc.},$$

we find

$$e_{11311}^2-f_{111}=0,\qquad e_{12312}^2-f_{121}=0,\qquad e_{21321}\cdot=0,\qquad e_{22322}\cdot=0,$$
$$e_{31331}^2-f_{313}=0,\qquad e_{32332}^2-f_{323}=0,\qquad e_{41341}\cdot=0,\qquad e_{42342}\cdot=0,$$

therefore

$$\lambda_{11} = \tfrac{1}{2}(f_{111} + e_{11311}), \qquad \lambda_{22} = \tfrac{1}{2}(f_{111} - e_{11311}),$$
$$\lambda_{33} = \tfrac{1}{2}(f_{121} + e_{12312}), \qquad \lambda_{44} = \tfrac{1}{2}(f_{111} - e_{12312}),$$
$$\lambda_{55} = f_{212}, \qquad \lambda_{56} = e_{21321},$$
$$\lambda_{65} = e_{22322}, \qquad \lambda_{66} = f_{222},$$
$$\lambda_{77} = \tfrac{1}{2}(f_{313} + e_{31331}), \qquad \lambda_{88} = \tfrac{1}{2}(f_{313} - e_{31331}),$$
$$\lambda_{99} = \tfrac{1}{2}(f_{323} + e_{32332}), \qquad \lambda_{aa} = \tfrac{1}{2}(f_{323} - e_{32332}),$$
$$\lambda_{bb} = f_{414}, \qquad \lambda_{bc} = e_{41341},$$
$$\lambda_{cb} = e_{42342}, \qquad \lambda_{cc} = f_{424}.$$

$$e_1 = \lambda_{11} + \lambda_{22} + \lambda_{33} + \lambda_{44} - \omega\lambda_{55} - \omega\lambda_{66} - \lambda_{77} - \lambda_{88} - \lambda_{99} - \lambda_{aa} + \omega\lambda_{bb} + \omega\lambda_{cc}.$$
$$e_2 = \lambda_{11} + \lambda_{22} - \lambda_{33} - \lambda_{44} + \omega\lambda_{55} - \omega\lambda_{66} + \lambda_{77} + \lambda_{88} - \lambda_{99} - \lambda_{aa} - \omega\lambda_{bb} + \omega\lambda_{cc}.$$
$$e_3 = \lambda_{11} - \lambda_{22} - \lambda_{33} + \lambda_{44} + \lambda_{56} + \lambda_{65} - \lambda_{77} + \lambda_{88} - \lambda_{99} + \lambda_{aa} + \lambda_{bc} + \lambda_{cb}.$$

PART 3. RELATION TO THE THEORIES OF GROUP DETERMINANTS, GROUP CHARACTERS, AND GROUPS DEFINED BY FINITE GROUPS.

1. We consider a group of order n with the n operators e_r $(r = 1, \cdots, n)$ and write the general number of its group-algebra in terms of the corresponding units e_r,

$$\phi = \Sigma x_r e_r.$$

If we also use for x_k the notation x_{ij} in case $e_i e_j = e_k$, and the notation $x_{ij^{-1}}$ in case $e_i e_j^{-1} = e_k$, we find by multiplying ϕ into each unit e_r $(r = 1, \cdots, n)$, that ϕ satisfies the equation:

$$f(\phi) \equiv \begin{vmatrix} x_{11^{-1}} - \phi, & x_{12^{-1}}, & \cdots, & x_{1n^{-1}} \\ x_{21^{-1}}, & x_{22^{-1}} - \phi, & \cdots, & x_{2n^{-1}} \\ \cdot & \cdot & \cdot & \cdot \\ x_{n1^{-1}}, & x_{n2^{-1}}, & \cdots, & x_{nn^{-1}} - \phi \end{vmatrix} = 0,$$

of degree n, the general equation of ϕ. The *characteristic* equation satisfied by ϕ is usually of much lower degree, whose factors all appear however in the equation $f(\phi) = 0$. The determinant $f(0)$ is the absolute term of the general equation of ϕ. This determinant may also be written, by shifting the rows and columns, in the form $\pm |x_{ij}| (i, j = 1, \cdots, n)$. The first form of the determinant is the *group-determinant* of FROBENIUS, * who studied the factorisation of

* Berliner Sitzungsberichte: *Über Gruppencharaktere*, 1896, pp. 985–1021; *Über die Primfactoren der Gruppendeterminante*, 1896, pp. 1343–1382; *Über die Darstellung der endlicher*

it and gave the theory of certain coefficients involved in its expansion which he called *group-characters.*

2. If we suppose that for every unit e we substitute its value in terms of the λ's we have $\phi = \Sigma a_{ijk}\lambda_{ijk}$. As shown above, the λ's appear in sets of μ_r^2. The coefficients a are linear homogeneous functions of the x's of § 1. The characteristic equation of ϕ will consist of the shear factors corresponding to the sets of λ's, each shear factor being a determinant and appearing to the first degree only, since there are no λ's of weight more than zero.* These shear factors appear in the general equation of ϕ, which is of degree n, to the same power as their width, since in the *frame*,† each factor of order μ_r appears μ_r times on the diagonal.‡ In previous discussions relating to this subject the general equation of ϕ has been treated,§ but no mention has been made of the reduced equation. It is readily seen that the determinant of ϕ is the product of the determinants of the shear factors, each to a degree equal to its width. The determinant factors are the *irreducible factors* of FROBENIUS.

To make the matter still clearer, let us consider the group of order 6, defined by the generators $e_1^3 = 1 = e_2^2$, $e_2 e_1 = e_1^2 e$. Let $\omega^3 = 1$. Having a dihedron group, we obtain as the six units

$$e_1 = \lambda_{11} + \lambda_{22} + \omega^2\lambda_{33} + \omega\lambda_{44}, \qquad e_2 = \lambda_{11} - \lambda_{22} + \lambda_{34} + \lambda_{43},$$
$$e_3 = \lambda_{11} + \lambda_{22} + \omega\lambda_{33} + \omega^2\lambda_{44}, \qquad e_4 = \lambda_{11} - \lambda_{22} + \omega^2\lambda_{34} + \omega\lambda_{43},$$
$$e_6 = \lambda_{11} + \lambda_{22} + \lambda_{33} + \lambda_{44}, \qquad e_5 = \lambda_{11} - \lambda_{22} + \omega\lambda_{34} + \omega_2\lambda_{43}.$$

Therefore

$$\begin{aligned}\phi = \lambda_{11}(x_1 + x_2 + x_3 + x_4 + x_5 + x_6) + \lambda_{22}(x_1 + x_3 + x_6 - x_2 - x_4 - x_5)\\ + \lambda_{33}(x_6 + \omega x_3 + \omega^2 x_1) + \lambda_{34}(x_2 + \omega x_5 + \omega^2 x_4)\\ + \lambda_{43}(x_2 + \omega x_4 + \omega^2 x_5) + \lambda_{44}(x_6 + \omega x_1 + \omega^2 x_3).\end{aligned}$$

The characteristic equation of ϕ is

$$(x_6 + x_1 + x_2 + x_3 + x_4 + x_5 - \phi)(x_6 + x_1 + x_3 - x_2 - x_4 - x_5 - \phi)$$
$$\begin{vmatrix} x_6 + \omega x_3 + \omega^2 x_1 - \phi & x_2 + \omega x_5 + \omega^2 x_4 \\ x_2 + \omega x_4 + \omega^2 x_5 & x_6 + \omega x_1 + \omega^2 x_3 - \phi \end{vmatrix} = 0.$$

Gruppen durch lineare Substitutionen, 1897, pp. 994–1015; 1899, pp. 482–500; *Über Relationen zwischen den Charakteren einer Gruppe und denen ihrer Untergruppen*, 1898, pp. 501–515; *Über die Composition der Charaktere einer Gruppe*, 1899, pp. 330–339; *Über die Charaktere der Symmetric und Alternirenden Gruppen*, 1900, pp. 516–534; 1901, pp. 303–315.

DICKSON, *Elementary Exposition of Frobenius' Theory of Group-Characters and Group-Determinants*, Annals of Mathematics, vol. 4 (1902), pp. 25–49.

* *Theory*, pp. 264, 275, 277.

† *Theory*, p. 265.

‡ *Theory*, pp. 275, 276.

§ See citations above.

The general equation has these same factors, the determinant however appearing to degree 2. The equation may be written also

$$f(\phi) = \begin{vmatrix} x_6 - \phi & x_1 & x_2 & x_3 & x_4 & x_5 \\ x_3 & x_6 - \phi & x_4 & x_1 & x_5 & x_2 \\ x_2 & x_5 & x_6 - \phi & x_4 & x_1 & x_3 \\ x_1 & x_3 & x_5 & x_6 - \phi & x_2 & x_4 \\ x_4 & x_2 & x_3 & x_5 & x_6 - \phi & x_1 \\ x_5 & x_4 & x_1 & x_2 & x_3 & x_6 - \phi \end{vmatrix} = 0,$$

the determinant being $f(0)$.

It is obvious that any matric algebra, whether group algebra or not, has a general equation which contains shear factors that belong to the characteristic equation each with an exponent equal to some order; but these exponents in the *characteristic* equation are all unity. For example, quaternions reduces to a form expressible in four λ's, having a characteristic equation of degree 2, and a general equation which is the square of the reduced. This property is due to the absence in the frame of the algebra of any terms off the main diagonal. That is, if λ_{rs0} appears, then λ_{sr0} also appears. This theorem may also be proved for group-algebras by proving that if ϕ is any given number of such an algebra, then if ψ be every number of the algebra, it is impossible that $\phi\psi$ can have only zero for a root for all values of ψ. This is equivalent to saying that if λ_{rs0} occur in ϕ, then λ_{sr0} must also occur in some ψ, and that λ_{rst}, $t \neq 0$, cannot occur. If we put $\phi = \Sigma x_r e_r$, $\psi = \Sigma y_s e_s$, e_t being one of n units, then we have

$$\phi\psi = \Sigma x_r y_s \cdot e_{rs} \qquad (e_{rs} = e_r \cdot e_s).$$

Now for $\phi\psi$ to have only zero roots for every value of y_r, the coefficients of the equation of $\phi\psi$ must all vanish, leaving $(\phi\psi)^n = 0$. But each coefficient reduces to a sum of products of determinants of the x's multiplied by corresponding determinants of the y's. Each such determinant in the x's must vanish then, the y's being arbitrary; that is, all minors of the determinant of ϕ must vanish. But the coefficient of $(\phi\psi)^{n-1}$ cannot thus vanish unless each x vanishes. Hence every expression ϕ has a multiplier which produces a term λ_{ii0}, and hence if λ_{rs0} is such a number ϕ, there must also be λ_{sr0} in the algebra, and it is a matric algebra.

3. Turning our attention to the equation of ϕ in its general form, we may write it, as any matrix equation may be written,

$$\phi^n - m_1\phi^{n-1} + m_2\phi^{n-2} \cdots \pm m_n = 0.$$

The coefficient m_n is the determinant of the matrix. The coefficients m_1, m_2,

$\cdots, m_n$, are respectively the sums of all minors of orders $1, 2, \cdots, n-1$, whose main diagonal is that of the determinant m_n. Thus, in the example above we have the following table of coefficients for the six numbers ϕ:

ϕ	m_1	m_2	m_3	m_4	m_5	m_6
e_0	6	15	20	15	6	1
e_1	0	0	2	0	0	1
e_2	0	3	0	3	0	1
e_3	0	0	2	0	0	1
e_4	0	3	0	3	0	1
e_5	0	3	0	3	0	1

Confining our attention to the coefficient m_1, we notice that each coefficient m_1 is the sum of what we may call *partial coefficients* of the nature of m_1, one from each quadrate. Thus $m_1 =$

	$\{\lambda_{11}\}$		$\{\lambda_{22}\}$		$\{\lambda_{33}, \lambda_{34}, \lambda_{43}, \lambda_{44}\}$	
e_0	1	$+$	1	$+$	$2 \cdot 2$	$= 6$,
e_1	1	$+$	1	$+$	$2(\omega^2 + \omega)$	$= 0$,
e_2	1	$-$	1	$+$	$2 \cdot 0$	$= 0$,
e_3	1	$+$	1	$+$	$2(\omega + \omega^2)$	$= 0$,
e_4	1	$-$	1	$+$	$2 \cdot 0$	$= 0$,
e_5	1	$-$	1	$+$	$2 \cdot 0$	$= 0$.

These *partial coefficients* from the different regions are the *group characters* of FROBENIUS.* It is to be noticed that they are the same for sets of operators, viz., for sets of conjugate operators. Thus for

	(11)	(22)	(34)
e_0	$m_1' = 1$,	$m_1'' = 1$,	$m_1''' = 2$,
e_1, e_3	$m_1' = 1$,	$m_1'' = 1$,	$m_1''' = \omega^2 + \omega = -1$,
e_2, e_4, e_5	$m_1' = 1$,	$m_1'' = -1$,	$m_1''' = 0$.

This is to be expected from the theory of matrices. For, in any case, if ϕ, χ belong to the same ground, ϕ and $\chi\phi\chi^{-1}$ have the same equation, and in each exclusive region have the same coefficients. This means of course that any

* Cf. DICKSON, *loc. cit.*

operator and its conjugates must have the same *characters*. In the example given,

$$e_3 = e_2 e_1 e_2^{-1}, \qquad e_4 = e_1^2 e_2 e_1^{-2}, \qquad e_5 = e_1 e_2 e_1^{-1}.$$

4. Within the limits of this paper it is impossible to re-develop the whole theory of group-characters, but it is evident that such development is quite possible on the basis of this theory, and indeed is quite simple. The relations of the coefficients $m_1, m_2, \cdots, m_n$ of any expression ϕ to those of $e_1, \cdots, e_n$ have been developed in papers read before this society some time ago.*

5. Reverting now to the equation

$$\phi = \Sigma x_r e_r \qquad (r = 1, \cdots, n),$$

it is obvious that if

$$\psi = \Sigma y_s e_s \qquad (s = 1, \cdots, n),$$

then

$$\phi\psi = \Sigma\Sigma x_{rs^{-1}} y_s e_r \qquad (r, s = 1, \cdots, n).$$

Consequently $\phi\psi$ belongs to the algebra, and if the coefficients of $\phi, \psi, \cdots$ run through all scalar values, real or imaginary, the expressions $\phi, \psi, \cdots$ form a continuous group. Since these equations are equivalent to a continuous group of linear homogeneous substitutions, this gives a basis for BURNSIDE'S theory of continuous groups defined by finite groups.† The theory may be developed from the preceding results.

6. BURNSIDE's theory of continuous groups defined by finite groups has been generalized by DICKSON,‡ in that the coefficients x_r, y_s, are taken as numbers (marks) in any field F whatsoever. The management of this problem develops the need of certain canonical forms which are essentially the units λ_{ij} above. If the coefficients of any linear associative algebra are restricted § to any field F, we may nevertheless develop the theorems of such algebra, and this is particularly true of the group-algebras belonging to such fields. In fact, the determination of the group-algebra of any finite group with respect to any field seems to be a necessary common basis for the diverse problems mentioned.

December 1, 1903.

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* Bulletin of the American Mathematical Society, vol. 5 (1899), p. 381. These papers are unpublished. It may be noted that FROBENIUS' method determines the width of the exclusive regions of λ's but does not determine $e_1, \cdots, e_n$ in terms of the λ's. This method determines λ_{ij0} in terms of $e_1, \cdots, e_n$, whence the inverse problem becomes easy.

† Proceedings of the London Mathematical Society, vol. 29 (1898), pp. 207–221, 546–565.

‡ *On the Group defined for ang given Field by the Multiplication Table of any Given Finite Group*, Transactions of the American Mathematical Society, vol. 3 (1902), pp. 285–301. Also certain special cases, Proceedings of the London Mathematical Society, vol. 35 (1902), pp. 68-80. Also *Groups defined for a General Field by the Rotation Groups*, University of Chicago Decennial Publications, vol. 9 (1902), pp. 35–51.

§ This generalization was first made by DICKSON : *Definitions of a Linear Associative Algebra by Independent Postulates*, Transactions of the American Mathematical Society, vol. 4 (1903), pp. 21–26.